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السلام عليكم ورحمة الله وبركاته .. وبعد ..

نفيدكم بقبول البحث المقدم من قبلكم للنشر بمجلة جامعة الملك عبد العزيز « العلوم » ،
تحت عنوان : Some Fixed Point Theorems for Subcompatible Maps
بعض نظريات النقطة الثابتة للرواسم المتسقة جزئياً

وسنوافيكم بالمسودة لتصحيحها واعادتها الينا بأسرع ما يمكن .

وتقبلوا أطيب تحياتي

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رئيس التحرير

أ.د. حمزة بن علي العجيل

Some Fixed Point Theorems For Subcompatible Maps [1]

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Abstract

Common fixed point and invariant approximation results are presented for subcompatible maps, a class of noncommuting maps, recently introduced in the literature. This work extends some well-known results, especially, those of Hussain and Khan (2003), Hussain and Rhoades (2006), Sahab, Khan and Sessa (1998) and Singh (1979).

1. Introduction and preliminaries

Let (E, τ) be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in I\}$ of seminorms on E is said to be an associated family of seminorms for τ if the family $\{\gamma U : \gamma > 0\}$, where $U = \cap_{i=1}^{n} U_{\alpha_i}$ and $U_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighborhoods of zero for τ. A family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E is called an augmented associated family for τ if $\{p_\alpha : \alpha \in I\}$ is an associated family with property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}$ for any $\alpha, \beta \in I$. The associated and augmented associated families of seminorms will be denoted by $A(\tau)$ and $A^*(\tau)$, respectively. It is well known that given a locally convex space (E, τ), there always exists a family $\{p_\alpha : \alpha \in I\}$ of seminorms defined on E such that $\{p_\alpha : \alpha \in I\} = A^*(\tau)$ (see [15]).

The following construction will be crucial. Suppose that M is τ-bounded subset of E. For this set M we can select a number $\lambda_\alpha > 0$ for each $\alpha \in I$ such that $M \subset \lambda_\alpha U_\alpha$ where $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Clearly, $B = \cap_\alpha \lambda_\alpha U_\alpha$ is τ-bounded, τ-closed absolutely convex and contains M. The linear span E_B of B in E is $\cup_{n=1}^{\infty} nB$. The Minkowski functional of B is a norm $\|\cdot\|_B$ on E_B. Thus $(E_B, \|\cdot\|_B)$ is a normed space with B as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_B$ for each $x \in E_B$ (see [15] and [22]).

[1] Key words: fixed point, subcompatible maps, compatible maps, invariant approximation.
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Let M be a subset of a locally convex space (E, τ). Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called I-Lipschitz if there exists $k \geq 0$ such that

$$p_\alpha(Tx - Ty) \leq kp_\alpha(Ix - Iy)$$

for any $x, y \in M$ and for all $p_\alpha \in A^*(\tau)$. If $k < 1$ (respectively, $k = 1$), then T is called an I-contraction (respectively, I-nonexpansive). A point $x \in M$ is a common fixed point of I and T if $x = Ix = Tx$. The set of fixed points of I is denoted by $F(I)$. The pair $\{I, T\}$ is called: (1) commuting if $TIx = ITx$ for all $x \in M$. (2) R-weakly commuting if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists $R > 0$ such that $p_\alpha(ITx - TIx) \leq Rp_\alpha(Ix - Tx)$. If $R = 1$, then the maps are called weakly commuting. (3) compatible, if for all $p_\alpha \in A^*(\tau)$, $\lim_n p_\alpha(TIx_n - ITx_n) = 0$ when $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in M.

Suppose that M is q-starshaped with $q \in F(I)$, we define $S_q(I, T) := \cup\{S(I, T_k) : 0 \leq k \leq 1\}$ where $T_k x = (1-k)q + kTx$ and $S(I, T_k) = \big\{\{x_n\} \subset M : \lim_n Ix_n = \lim_n T_k x_n = t \in M \Rightarrow \lim_n p_\alpha(IT_k x_n - T_k I x_n) = 0\big\}$, for all $p_\alpha \in A^*(\tau)$. Then I and T are called: (4) subcompatible if

$$\lim_n p_\alpha(ITx_n - TIx_n) = 0$$

for all sequences $\{x_n\} \in S_q(I, T)$, (5) R-subcommuting on M, if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq \frac{R}{k} p_\alpha(((1-k)q + kTx) - Ix)$ for each $k \in (0, 1]$. If $R = 1$, then the maps are called 1-subcommuting; (6) R-subweakly commuting on M, if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq Rd_{p_\alpha}(Ix, [q, Tx])$, where $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$.

Note that subcompatible maps are compatible [1] but the converse does not hold, in general. Recall that weakly commuting maps are compatible but converse fails to hold.

If $u \in E, M \subseteq E$, then we define the set $P_M(u)$ of best M-approximants to u as follows:

$$P_M(u) = \{y \in M : p_\alpha(y - u) = d_{p_\alpha}(u, M), \forall p_\alpha \in A^*(\tau)\},$$

where

$$d_{p_\alpha}(u, M) = \inf\{p_\alpha(x - u) : x \in M\}.$$

A mapping $T : M \to M$ is called demiclosed at 0 if for every sequence $\{x_n\} \in M$ such that $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to 0, we have $Tx = 0$.

In [4], Fisher and Sessa obtained the following generalization of a theorem of Gregus [5].

Theorem 1. *Let T and I be two weakly commuting mapping of a closed convex subset C of a Banach spaces X into itself satisfying the inequality*

$$\|Tx - Ty\| \leq a\|Ix - Iy\| + (1-a)\max\{\|Tx - Ix\|, \|Tx - Ix\|\},$$

for all $x, y \in C$, where $a \in (0,1)$. If I is linear and nonexpansive on C and $T(C) \subseteq I(C)$, then T and I have a unique common fixed point in C.

In 1993, Jungck and Rhoades [11] obtained the following theorem.

Theorem 2. *Let T and I be compatible self maps of C, a closed convex subset of a Banach space X, satisfying:*

$$\begin{aligned}\|Tx - Ty\| \leq & \alpha\|Ix - Iy\| + \beta\max\{\|Tx - Ix\|, \|Ty - Iy\|\} \\ & + \gamma\max\{\|Ix - Iy\|, \|Tx - Ix\|, \|Ty - Iy\|\},\end{aligned}$$

for all $x, y \in C$, where $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = 1$. If I is linear and continuous in C and $T(C) \subseteq I(C)$, then T and I have a unique common fixed point.

In this paper, we first prove that Theorems 1-2 can be extended to the setup of a Hausdorff locally convex space. As application, common fixed point and invariant approximation results for subcompatible maps are derived. Our results extend and unify the work of Baskaran and Subrahhmanyam [2], Brosowski [3], Hussain and Khan [6], Jungck and Sessa [12], Khan and Hussain [13], Pathak, Cho, and Kang [17], Sahab, Khan, and Sessa [18], Shahzad [20] and Singh [21]. For recent results, on common fixed point and approximations, we refere the reader to [8, 9, 12, 13].

2. Main results

Lemma 1. *Let T and I be compatible selfmaps of a τ-bounded subset M of a Hausdorff locally convex space (E, τ). Then T and I are compatible on M with respect to $\|\cdot\|_B$.*

Proof. By hypothesis, there is a sequence $\{x_n\}$ such that $\lim_n p_\alpha(ITx_n - ITx_n) = 0$ for each $p_\alpha \in A^*(\tau)$, whenever $\lim_{n\to\infty} p_\alpha(Tx_n - t) = 0 = \lim_{n\to\infty} p_\alpha(Ix_n - t)$ for some $t \in M$. Taking supremum on both sides, we get

$$\sup_\alpha \lim_{n\to\infty} p_\alpha(\frac{ITx_n - TIx_n}{\lambda_\alpha}) = \sup_\alpha p_\alpha(\frac{0}{\lambda_\alpha})$$

This implies that

$$\lim_{n\to\infty} \sup_\alpha p_\alpha(\frac{ITx_n - TIx_n}{\lambda_\alpha}) = 0$$

whenever,

$$\lim_{n\to\infty} \sup_\alpha p_\alpha(\frac{Tx_n - t}{\lambda_\alpha}) = 0 = \lim_{n\to\infty} \sup_\alpha p_\alpha(\frac{Ix_n - t}{\lambda_\alpha}).$$

Hence, $\lim_{n\to\infty} \|ITx_n - TIx_n\|_B = 0$, whenever $\lim_{n\to\infty} \|Tx_n - t\|_B = 0 = \lim_{n\to\infty} \|Ix_n - t\|_B$ as desired. □

The next theorem generalizes Theorems 1-2.

Theorem 3. *Let M be a nonempty τ-bounded, τ-complete, and convex subset of a Hausdorff locally convex space (E, τ) and T and I be compatible selfmaps of M satisfying the inequality*

$$\begin{aligned} p_\alpha(Tx - Ty) \leq a p_\alpha(Ix - Iy) &+ b \max\{p_\alpha(Tx - Ix),\, p_\alpha(Ty - Iy)\} \\ &+ c \max\{p_\alpha(Ix - Iy), p_\alpha(Tx - Ix),\, p_\alpha(Ty - Iy)\}, \end{aligned} \tag{1}$$

for all $x, y \in M$ and for all $p_\alpha \in A^*(\tau)$, where $a, b, c > 0$ and $a+b+c = 1$. If I is linear and nonexpansive on M and $T(M) \subseteq I(M)$, then T and I have a unique common fixed point.

Proof. Since M is τ-complete, it follows that $(E_B, \|\cdot\|_B)$ is a Banach space and M is complete in it. By Lemma 1, T and I are compatible with respect to $\|\cdot\|_B$ on M. From (1) we obtain for $x, y \in M$,

$$\begin{aligned} \sup_\alpha p_\alpha\left(\frac{Tx - Ty}{\lambda_\alpha}\right) &\leq \sup_\alpha p_\alpha\left(\frac{Ix - Iy}{\lambda_\alpha}\right) \\ &+ a \max\left\{\sup_\alpha p_\alpha(\frac{Tx - Ix}{\lambda_\alpha}), \sup_\alpha p_\alpha(\frac{Ty - Iy}{\lambda_\alpha})\right\} \\ &+ b \max\left\{\sup_\alpha p_\alpha(\frac{Ix - Iy}{\lambda_\alpha}), \sup_\alpha p_\alpha(\frac{Tx - Ix}{\lambda_\alpha}), \sup_\alpha p_\alpha(\frac{Ty - Iy}{\lambda_\alpha})\right\}. \end{aligned}$$

Thus

$$\begin{aligned} \|Tx - Ty\|_B \leq a \|Ix - Iy\|_B &+ b \max\{\|Tx - Ix\|_B, \|Ty - Iy\|_B\} \\ &+ c \max\{\|Ix - Iy\|_B, \|Tx - Ix\|_B, \|Ty - Iy\|_B\}. \end{aligned}$$

It can be shown easily that I is $\|\cdot\|_B$−nonexpansive on M. A comparison of our hypothesis with that of Theorem 2 tells that we can apply Theorem 2 to M as a subset of $(E_B, \|\cdot\|_B)$ to conclude that there exists a unique $a \in M$ such that $a = Ia = Ta$. □

The following theorem generalizes Theorem 3 in [20] and corresponding result in [8] to a more general class of functions.

Theorem 4. *Let T and I be selfmaps of a convex subset M of a Hausdorff locally convex space (E,τ). Suppose that I is nonexpansive and linear on M, $q \in F(I)$ and $T(M) \subseteq I(M)$. Assume that the pair $\{I,T\}$ is subcompatible and satisfies, for all $p_\alpha \in A^*(\tau)$, $x,y \in M$, and for all $k \in (0,1)$ with $0 < a,b < 1$, $a+b=1$*

$$P_\alpha(Tx-Ty) \leq p_\alpha(Ix-Iy) + a(\frac{1-k}{k}) \max\Big\{d_{p_\alpha}(Ix,[q,Tx]),\, d_{p_\alpha}(Iy,[q,Ty])\Big\} + b(\frac{1-k}{k}) \max\Big\{d_{p_\alpha}(Ix,[q,Iy]),\, d_{p_\alpha}(Ix,[q,Tx]), d_{p_\alpha}(Iy,[q,Ty])\Big\}. \quad (2)$$

Then I and T have a common fixed point in M provided one of the following conditions holds:

(i) M is τ-compact and T is continuous.

(ii) M is weakly compact in (E,τ), I is weakly continuous and $I-T$ is demiclosed at 0.

Proof. Let $\{k_n\}$ be a sequence of real numbers such that $0 < k_n < 1$ and $\lim_n k_n = 1$. Define for each $n \in \mathbf{N}$, a mapping $T_n : M \to M$ by

$$T_n(x) = k_n Tx + (1-k_n)q,$$

for some q and all $x \in M$. Then for each n, $T_n(M) \subseteq I(M)$, since I is linear, $Iq = q$ and $T(M) \subseteq I(M)$.

Since, the pair $\{I,T\}$ is subcompatible, for any $\{x_m\} \subset M$ with $\lim_m Ix_m = \lim_m T_n x_m = t \in M$,we have

$$\begin{aligned}\lim_m p_\alpha(T_n I x_m - I T_n x_m) &= k_n \lim_m p_\alpha(TIx_m - ITx_m) \\ &= 0.\end{aligned}$$

Thus, the pair $\{I, T_n\}$ is compatible on M for each n. We obtain from (2),

$$\begin{aligned}p_\alpha(T_n x - T_n y) &= k_n p_\alpha(Tx - Ty) \\ &\leq k_n p_\alpha(Ix - Iy) + a(1-k_n)\max\{p_\alpha(Ix - T_n x),\, p_\alpha(Iy - T_n y)\} \\ &\quad + b(1-k_n)\max\{p_\alpha(Ix - Iy),\, p_\alpha(Ix - T_n x),\, p_\alpha(Iy - T_n y)\}\end{aligned}$$

for each $x,y \in M$ and for all $p_\alpha \in A^*(\tau)$, $0 < k_n < 1$. Note that $k_n + a(1-k_n) + b(1-k_n) = 1$ for all n.

(i) M being τ-compact is τ-bounded and τ-complete. Thus by Theorem 3, for each $n \geq 1$, there exists an $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. Now the τ-compactness of M ensures that $\{x_n\}$ has a convergent subsequence $\{x_j\}$ which converges to a point $x_0 \in M$. Since

$$x_j = T_j x_j = k_j T x_j + (1-k_j)$$

and T is continuous, so we have, as $j \to \infty, Tx_0 = x_0$. The continuity of I implies that

$$Ix_0 = I\big(\lim_j x_j\big) - \lim_j I(x_j) = \lim_j x_j = x_0.$$

(ii) Weakly compact sets in (E, τ) are τ-bounded and τ-complete so again by Theorem 3, T_n and I have a common fixed point x_n in M for each n. The set M is weakly compact so there is a subsequence $\{x_j\}$ of $\{x_n\}$ converging weakly to some $y \in M$. The map I being weakly continuous gives that $Iy = y$. Now

$$x_j = I(x_j) = T_j(x_j) = k_j T x_j + (1 - k_j) q$$

implies that $Ix_j - Tx_j = (1-k_j)[q - Tx_j] \to 0$ as $j \to \infty$. The demiclosedness of $I - T$ at 0 implies that $(I - T)(y) = 0$. Hence $Iy = Ty = y$.

□

As an application of Theorem 4, we establish the following result in best approximation theory which extends and improves the corresponding results in [2, 3, 6, 12, 13, 14, 17, 18, 20, 21]

Theorem 5. *Let T and I be selfmaps of a Hausdorff locally convex space (E, τ) and M a subset of E such that $T(\partial M) \subseteq M$, where ∂M denotes boundary of M and $u \in F(T) \cap F(I)$. If $P_M(u)$ is nonempty convex, $q \in F(I)$, I is nonexpansive and linear on $P_M(u)$ and $I(P_M(u)) = P_M(u)$. Suppose that the pair $\{I, T\}$ is subcompatible on $P_M(u)$ and satisfies, for all $x \in P_M(u) \cup \{u\}$, $p_\alpha \in A^*(\tau), k \in (0, 1)$,*

$$p_\alpha(Tx - Ty) \leq \begin{cases} p_\alpha(Ix - Iu) & \text{if} \quad y = u; \\ \Lambda(x, y) & \text{if} \quad y \in P_M(u) \end{cases} \tag{3}$$

where

$$\begin{aligned}\Lambda(x, y) = {} & p_\alpha(Ix - Iy) + a\Big(\frac{1-k}{k}\Big) \max\Big\{d_{p_\alpha}(Ix, [q, Tx]), d_{p_\alpha}(Iy, [q, Ty])\Big\} \\ & + b\Big(\frac{1-k}{k}\Big) \max\Big\{d_{p_\alpha}(Ix, [q, Iy]), d_{p_\alpha}(Ix, [q, Tx]),\ d_{p_\alpha}(Iy, [q, Ty])\Big\}.\end{aligned}$$

Then $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$, provided one of the following conditions holds:

(i) $P_M(u)$ is τ-compact and T is continuous.

(ii) $P_M(u)$ is weakly compact in (E, τ), I is weakly continuous and $I - T$ is demiclosed at 0.

Proof. Let $y \in P_M(u)$. Then $Iy \in P_M(u)$, since $I(P_M(u)) = P_M(u)$. Further, if $y \in \partial M$ then $Iy \in M$ for $T(\partial M) \subseteq M$. Also since $Ix \in P_M(u), u \in F(T) \cap F(I)$ and I and T satisfy (3), we have

$$p_\alpha(Tx - u) = p_\alpha(Tx - Tu) \leq p_\alpha(Ix - Iu) = p_\alpha(Ix - u) = d_{p_\alpha}(u, M),$$

for each $p_a \in A^*(\tau)$. Thus $Tx \in P_M(u)$ which implies that T maps $P_M(u)$ into itself and the conclusion follows from Theorem 4. □

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