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Similarity Analysis for the Gravitational Field

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ABSTRACT. We review stationary axially symmetrical empty space-times possessing similarity property. The solutions which we obtain generalize the solutions of Lewis and van Stockum. The potentials and the null-tetrad components of the Riemann tensor are computed. It is concluded that rotating field solution, in an unrestricted sense, is incompatible with the concept of self-similarity.

Introduction

The topic of similarity solutions constitutes a significant amount of work in the theory of partial differential equations, most particularly when it happens that exact solutions to non-linear equations remain elusive for long times. In reality, similarity methods apply to any theory that the fields-streamlines are expressed in scalar potentials, while the presence of multi variables renders exact solutions almost hopeless. As a prototype example, we mention the field of classical hydrodynamics, where similarity solutions provide the basic feedback to any advancement taking place in this particular topic.

In general, relativity similarity solutions to Einstein equations have not been studied systematically, except in some particular cases of cosmological interest[1] and in connection with Ernst equation[2,3]. Some exact solutions well-known for decades are in fact nothing other than similarity solutions, although they all arise in different contexts. In this category, we can cite the solutions of Lewis[4], van Stockum[5] and plane waves in the Rosen form.

In this paper, we don't intend a detailed exposition of the topic, but rather we restrict ourselves to the stationary vacuum space-times alone. The problem is shown to reduce in this special case to a single master equation; the one dimensional Liouville's equation. From the solutions of this single equation, we prove that Lewis and van Stockum solutions can naturally be generalized.

In a previous paper[6], we had introduced a method of reduction for certain classes of partial differential equations (pde) into ordinary ones that are readily solvable. The method, together with its limitations, manifests itself best in problems where the pde systems are derivable from a purely kinetic Lagrangian density. Further, the Lagrangian density was defined *via* a geometrical method, known as the harmonic mappings of Riemannian manifolds[7], so that covariance of the formalism becomes manifest. Reduction in the dependent variables is equally as important as the reduction in the independent variables, however, the former should be taken cautiously in the presence of a non-Abelian gauge group acting in the theory. Since we are interested in non-Abelian self-similar solutions, then, reduction to single dependent functions reduces the solution to an Abelian sub-group automatically.

Gravitation has great impact in our daily life, ranging from the modest effect in growing of plants to the very sophisticated space defence programs by its deflection of the radar beams. Although our familiarity with this weakest force of nature is such deep rooted, a complete understanding of it remains elusive yet. We maintain the belief, therefore, that any minor contribution in the field of gravitation will have direct impact on empirical results within the coming decades.

Stationary symmetrical gravitational fields admitting two killing vectors, one space-like and one time-like, are described by the canonical line element.

$$ds^2 = e^{2\psi}(dt - \omega d\phi)^2 - e^{-2\psi}\left[e^{2\gamma}(d\rho^2 + dz^2) + \lambda^2 d\phi^2\right] \tag{1}$$

where all the metric functions depend at most on ρ and z. The vacuum equations, $R_{\mu\nu} = 0$, (*i.e.* the Ricci tensor vanishes) are

$$\psi_{\rho\rho} + \psi_{zz} + \frac{1}{\lambda}(\psi_\rho\lambda_\rho + \psi_z\lambda_z) + \frac{1}{2\lambda^2}(\omega_\rho^2 + \omega_z^2)\,e^{4\psi} = 0 \tag{2}$$

$$\omega_{\rho\rho} + \omega_{zz} - \frac{1}{\lambda}(\omega_\rho\lambda_\rho + \omega_z\lambda_z) + 4(\omega_\rho\psi_\rho + \omega_z\psi_z) = 0 \tag{3}$$

$$\gamma_{\rho\rho} + \gamma_{zz} + \psi_\rho^2 + \psi_z^2 + \frac{e^{4\psi}}{4\lambda^2}(\omega_\rho^2 + \omega_z^2) = 0 \tag{4}$$

$$\lambda_{\rho\rho} + \lambda_{zz} = 0 \tag{5}$$

The variational principle that yields these equations is provided by the Lagrangian

$$L = 4(\gamma_\rho\lambda_\rho + \gamma_z\lambda_z) - 4\lambda(\psi_\rho^2 + \psi_z^2) + \frac{e^{4\psi}}{\lambda}(\omega_\rho^2 + \omega_z^2) \tag{6}$$

We shall proceed now by deviating from the usual trend of fixing the metric function λ equal to ρ. Although $\lambda = \rho$, is considered to be a requirement of axial symmetry and asymptotic flatness, we shall be violating these conditions by considering λ as a general function of ρ and z restricted only by equation (5). The structure of the foregoing Einstein equations suggests that if the similarity variable is chosen as a harmonic function, then the equations simplify to a great extent. Being prompted by this observation, we consider our metric functions ψ, ω and γ to be only the function of the harmonic λ.

The equations (2-4) reduce then into

$$\psi'' + \frac{1}{\lambda}\,\psi' + \frac{\omega'^2}{2\lambda^2}\,e^{4\psi} = 0 \tag{7}$$

$$\omega'' - \frac{1}{\lambda}\,\omega' + 4\psi'\,\omega' = 0 \tag{8}$$

$$\gamma'' + \psi'^2 + \frac{\omega'^2}{4\lambda^2}\,e^{4\psi} = 0 \tag{9}$$

where a prime denotes $\frac{d}{d\lambda}$. We readily observe that eq. (8) is integrated once to yield the expression

$$\omega' = k\,\lambda\,e^{-4\psi} \tag{10}$$

where k is an arbitrary constant of integration. Further, the eq. (9) admits the integrability condition

$$\gamma' = \lambda\,(\psi'^2 - \frac{k^2}{4}\,e^{-4\psi}) \tag{11}$$

The ψ equation takes the form

$$\psi'' + \frac{1}{\lambda}\,\psi' + \frac{k^2}{2}\,e^{-4\psi} = 0 \tag{12}$$

Defining, $Z = e^{-2\psi}$, transforms this equation into

$$Z'' = \frac{Z'^2}{Z} - \frac{1}{\lambda}\,Z' + k^2\,Z^3 \tag{13}$$

which is identified as one of the degenerate forms of a third Painleve's transcendental equation. This form, however, is not much impressive, therefore, we purse an alternative transformation. Introducing

$$\lambda = e^x$$
$$\tag{14}$$
$$\Phi = x - 2\psi$$

transforms eq. (12) into the one dimensional Liouville's equation,

$$\Phi_{xx} = k^2\,e^{2\Phi} \tag{15}$$

Corresponding to each distinct solution of this ordinary differential equation, we construct a different similarity metric given by eq. (1).

Note, however, that this equation may admit solutions which violate the signature of space-time and, therefore, should be discarded. The following three solutions are the only significant solutions at our disposal.

i) $$e^{\Phi} = e^{\alpha x}\,(1 - \frac{k^2}{4\alpha^2}\,e^{2\alpha x}\,)^{-1} \tag{16}$$

where α is an arbitrary real constant ($\neq 0$). The corresponding line element is

$$ds^2 = (\lambda^{1-\alpha} - \frac{k^2}{4\alpha^2}\lambda^{1+\alpha})\, dt^2 - \frac{k}{\alpha}\lambda^{1+\alpha}\, dt\, d\phi -$$

$$- \lambda^{\frac{(\alpha^2-1)}{2}} (d\rho^2 + dz^2) - \lambda^{1+\alpha}\, d\phi^2 \qquad (17)$$

ii) $e^{-\Phi} = \cos\,[\alpha\, \ln\,(\frac{x}{a})]$ (18)

where a is another constant such that the argument of ln will be positive. The line element becomes

$$ds^2 = \lambda\, \cos\,(\, k\, \ln\, \frac{\lambda}{a}\,)\, dt^2 - 2\lambda\, \sin\,(\, k\, \ln\, \frac{\lambda}{a}\,)\, dt\, d\phi$$

$$- \lambda^{-(\frac{1+\alpha^2}{2})} (d\rho^2 + dz^2) - \lambda\, \cos\,(\, k\, \ln\, \frac{\lambda}{a}\,)\, d\phi^2\,. \qquad (19)$$

iii) $e^{-\Phi} = 1 + kx,$ (20)

which yields the line element,

$$ds^2 = \lambda\,(1 + k\, \ln\, \lambda)\, dt^2 - 2\,\lambda\, dt\, d\phi - \lambda^{-\frac{1}{2}}\, (d\rho^2 + dz^2) \qquad (21)$$

For the simplest choice $\lambda = \rho$, the metrics (17) and (19) reduce to the ones given by Lewis and (21) to van Stockum solutions, respectively. For $k = 0$ (*i.e.* no rigid rotation), the metric (17) reduces to the metric of Levi-Civita[8], which for the special parameters, $\alpha = \pm 1$, becomes the flat metric. The admissible ranges of the λ function are restricted severely by the existence of horizons in the above metrics.

Obviously, from eq. (5), a special class of function is provided by, $\lambda = Re\, f(\rho + iz)$, where f is an analytic function in the (ρ, z) plane. Also, the separable forms of λ sought in the form, $\lambda = \rho\, R(\rho)\, Z(z)$, are given by, $\lambda = \rho\,(\frac{\sin \omega\rho}{\omega\rho})\, e^{\omega z}$, where ω is a separation constant. The condition $\lambda = \rho$, is recovered in the limit $\omega \rightarrow 0$.

Finally, the metrics given above may be important in connection with the spinning cosmic strings that have been proposed recently in the quantization of gravity[9]. To our knowledge these metrics were absent in the literature.

Detailed Analysis of The Solutions

In the foregoing section, we have obtained the solutions for self-similar gravitational fields in stationary axially symmetric geometry, and we proceed now to derive the physical implications of our model. Gravitation, in the language of tensors and spinors provide a rich structure described by a multitude of complex potentials. In order to obtain physical results, one has to go through the tedious task of computing all non-vanishing potentials and other related elements. Our approach will be the one due to Newman and Penrose (NP)[10], which is considered to be equivalent to the Einstein's tensor formulation of general relativity. Consider a general class of line element.

$$ds^2 = F^2\,dt^2 - 2H\,dt\,d\phi - G^2\,d\phi^2 - e^{2(\gamma-\psi)}\,(d\rho^2 + dz^2) \tag{22}$$

where the metric functions, F, G, H, ψ and γ are functions of ρ and z only. As a matter of fact, this form of the line element constitutes a good starting point to study more general fields when the time dependence is incorporated as well. However, in this article we shall suppress any dependence on time and study the much simpler case. The NP null-tetrad in which we compute the potentials is given by

$$\begin{aligned}
\sqrt{2}\;l_\mu &= F\,\delta^t_\mu + \frac{K-H}{F}\,\delta^\phi_\mu \;, \\
\sqrt{2}\;n_\mu &= F\,\delta^t_\mu - \frac{K+H}{F}\,\delta^\phi_\mu \;, \\
\sqrt{2}\;m_\mu &= e^{\gamma-\psi}\,(\delta^\rho_\mu + i\,\delta^z_\mu) \;.
\end{aligned} \tag{23}$$

The metric functions here are related to the ones of the line element (1) by

$$\begin{aligned}
F &= e^{\psi},\; H = \omega e^{2\psi},\; G^2 = \lambda^2 e^{-2\psi} - \omega^2 e^{2\psi} \\
K^2 &= F^2G^2 + H^2 = \lambda^{2'} \;.
\end{aligned} \tag{24}$$

The non-vanishing spin coefficients (= complex potentials), α, β, τ, π, κ and ν are given as follow:

$$\begin{aligned}
2\sqrt{2}\;e^{\gamma-\psi}\,\alpha &= (\gamma-\psi)_\rho - i\,(\gamma-\psi)_z + \frac{e^{2\psi}}{2\lambda}\,(\omega_\rho - i\omega_z) \;, \\
2\sqrt{2}\;e^{\gamma-\psi}\,\beta &= (\psi-\gamma)_\rho + i\,(\psi-\gamma)_z + \frac{e^{2\psi}}{2\lambda}\,(\omega_\rho + i\omega_z) \;, \\
2\sqrt{2}\;e^{\gamma-\psi}\,\tau &= (\ln\lambda)_\rho + i\,(\ln\lambda)_z \;, \\
2\sqrt{2}\;e^{\gamma-\psi}\,\pi &= -(\ln\lambda)_\rho + i\,(\ln\lambda)_z \;, \\
2\sqrt{2}\;e^{\gamma-\psi}\,\kappa &= -(\ln\lambda)_\rho - i\,(\ln\lambda)_z + 2\,(\psi_\rho + i\psi_z) + \frac{e^{2\psi}}{\lambda}\,(\omega_\rho + i\omega_z) \;, \\
2\sqrt{2}\;e^{\gamma-\psi}\,\nu &= (\ln\lambda)_\rho - i\,(\ln\lambda)_z - 2(\psi_\rho - i\psi_z) + \frac{e^{2\psi}}{\lambda}\,(\omega_\rho - i\omega_z) \;.
\end{aligned} \tag{25}$$

All these components correspond to the single potential function of the classical Newtonian gravitation since it has been replaced by the Einstein's theory of gravitation.

Since the Weyl components of the Riemann tensor are rather tedious compared to the potentials, we shall be satisfied only by their self-similar counterparts that simplifies the expressions to great extent. In the null-tetrad (23) only ψ_2, ψ_0 and ψ_4 are nonzero and are given as

$$2\, e^{2(\gamma-\psi)}\, \psi_2 = (\lambda_\rho^2 + \lambda_z^2)\, \left(\frac{\psi'}{\lambda} - \psi'^2 + \frac{e^{4\psi}}{4\lambda^2}\, \omega'^2\right)\,, \tag{26}$$

$$\begin{aligned} 2\, e^{2(\gamma-\psi)}\, \psi_0 = {} & \left(\psi' - \frac{1}{2\lambda} + \frac{e^{2\psi}}{2\lambda}\, \omega'\right) \{\lambda_{\rho\rho} - \lambda_{zz} + 2i\lambda_{\rho z} \\ & + 2(\psi' - \gamma')\, (\lambda_\rho + i\lambda_z)^2\}\,, \end{aligned} \tag{27}$$

$$\begin{aligned} 2\, e^{2(\gamma-\psi)}\, \psi_4 = {} & \left(\psi' - \frac{1}{2\lambda} - \frac{e^{2\psi}}{2\lambda}\, \omega'\right) \{\lambda_{\rho\rho} - \lambda_{zz} - 2i\lambda_{\rho z} + \\ & + 2\, (\psi' - \gamma')\, (\lambda_\rho - i\lambda_z)^2\}\,, \end{aligned} \tag{28}$$

where a prime implies $\frac{d}{d\lambda}$.

Although the null-tetrad (23) serves to describe all the solutions (17,19,21) pretty well, there are some benefits in describing (21) in an alternative tetrad. Since the generic form of the line element in this case is

$$ds^2 = e^{2\psi}\, dt\, (dt - 2\omega\, d\phi) - e^{2(\gamma-\psi)}\, (d\rho^2 + dz^2)\,, \tag{29}$$

we choose the following null-tetrad,

$$\begin{aligned} l_\mu &= \frac{1}{2}\, e^{2\psi}\, \delta_\mu^t\,, \\ n_\mu &= \delta_\mu^t - 2\omega\, \delta_\mu^\phi\,, \\ \sqrt{2}\, m_\mu &= e^{\gamma-\psi}\, (\delta_\mu^\rho + i\delta_\mu^z)\,. \end{aligned} \tag{30}$$

Non-vanishing spin coefficients and the Weyl curvature components are tabulated below (note that the self-similarity requirement has not been imposed)

$$\begin{aligned} 2\sqrt{2}\, e^{\gamma-\psi}\, \alpha &= (\ln \omega^{1/2}\, e^{\gamma-2\psi})_\rho - i\, (\ln \omega^{1/2}\, e^{\gamma-2\psi})_z\,, \\ 2\sqrt{2}\, e^{\gamma-\psi}\, \beta &= (\ln \omega^{1/2}\, e^{-\gamma})_\rho + i\, (\ln \omega^{1/2}\, e^{-\gamma})_z\,, \\ \sqrt{2}\, e^{\gamma-\psi}\, \pi &= -\, (\ln \omega^{1/2} e^{\psi})_\rho + i\, (\ln \omega^{1/2}\, e^{\psi})_z\,, \\ \sqrt{2}\, e^{\gamma-\psi}\, \tau &= (\ln \omega^{1/2} e^{\psi})_\rho + i\, (\ln \omega^{1/2}\, e^{\psi})_z\,, \\ \frac{1}{\sqrt{2}}\, e^{\gamma+\psi}\, \nu &= (\ln \omega)_\rho - i\, (\ln \omega)_z\,, \\ \psi_1 = \psi_3 &= \psi_0 = 0. \end{aligned} \tag{31}$$

$$-2\, e^{2(\gamma-\psi)}\, \psi_2 = \omega^{-1/2}\, e^{-\psi}\, [(\omega^{1/2}\, e^{\psi})_{\rho\rho} + (\omega^{1/2}\, e^{\psi})_{zz}]\,, \tag{32}$$

$$\begin{aligned} e^{2\gamma}\, \psi_4 = {} & (\ln \omega)_{zz} - (\ln \omega)_{\rho\rho} + 2i(\ln \omega)_{\rho z} + \\ & + 2\, [(\ln \omega)_\rho - i\, (\ln \omega)_z]\, [(\gamma - 2\psi)_\rho - i\, (\gamma - 2\psi)_z]\,. \end{aligned} \tag{33}$$

Upon imposing the self-similarity condition, ψ_2 and ψ_4 simplify into

$$\psi_2 = -\frac{1}{2}\,[(\lambda^{1/2})_{\rho\rho} + (\lambda^{1/2})_{zz}]\ , \tag{34}$$

$$\psi_4 = \frac{k\,\lambda^{-3/2}}{(1 + k\ln\lambda)^2}\,\{\lambda_{\rho\rho} - \lambda_{zz} - 2i\lambda_{\rho z} + \frac{2}{\lambda}\,(\lambda_\rho - i\lambda_z)^2\}\ . \tag{35}$$

If the choice $k = 0$ is made, it can be seen that only ψ_2 survives and the resulting space becomes type-D. From the expression of ψ_4 in (33) it is also observed that the solution does not possess $\omega = 0$ limit. In fact, $\omega = 0$ leads to a degenerate space-time and, therefore, should be discarded. The solution, $\omega = \dfrac{1}{1 + k\ln\lambda}$, also reveals that for finite λ ($\lambda \neq 0$), ω does not become zero. The solutions (21) and (23) generalize the fields of sources in rigid rotation.

No matter which choice of null-tetrad basis is made in the space-time manifold, ψ_2 emerges real and rejects any complex components. The important conclusion to be drawn out from this result is that self-similarity nature does not yield a rotating field that resembles to the rotation given by the Kerr[11] metric.

The complex curvature components of all the above solutions arise in connection with the z-dependence. If the metric is frozen in the z-direction, then, all components will be real.

We would like to check now, whether the solutions we have obtained are significant or by some coordinate transformations they reduce to the previously well-known Lewis and van Stockum solutions. To this end, we define new coordinates by

$$\rho' + iz' = f(\rho + iz) \tag{36}$$

where f is an analytic function of this argument. One observes that $\rho' = Re\,f(\rho + iz) = \lambda$ is easily achieved. (Recall that a special class of λ was given as the real part of an analytic function). However, we have

$$|\,d\rho' + idz'\,| \neq |\,d\rho + idz\,|$$

meaning that the solutions are different. In other words, the similarity character does not remain invariant under the transformation (36), but admittedly it maps our solutions to some other solutions obtained by other means in different contexts.

Finally, in order to discuss the singular points of our space-time mainfolds, we have to substitute the explicit forms of the λ function into ψ_2, ψ_0 and ψ_4. For instance, the simplest possible choice, in which $\omega = 0$ and $\lambda = \rho$, which recovers the Levi-Civita metric as a degenerate form of solution (17), diverges at the origin, $\rho = 0$.

Conclusion

Our initial motive in conducting this research was to explore to what extent a *general solution* of stationary axially symmetrical vacuum fields can be obtained by employing similarity variables exhaustively. We have observed, as a result, that genuinely rotating fields can not be obtained by this method, leaving thus the Kerr metric once more unrivalled in this respect. Although in hydrodynamics genuine rotation arises albeit the self-similarity requirement, it seems that the complete analogy with the gravitational field breaks down. Our procedure can naturally be extended to the case of space-times admitting two space-like killing vectors. In this latter case, polarization of the gravitational waves play the similar role of rotation of our model presented in this paper.

Another conclusion to be drawn out is that in order to admit stationary twist, we must have $\psi_2 = 0$ with $Re\ \psi_0 \neq 0 \neq Im\ \psi_0$, (or the same conditions for ψ_4). This is not satisfied by any of our solutions, therefore, such a twist solution does not exist.

An interesting study consisting of a detailed analysis of similarity black-holes in gravitation will be our objective in a later communication.

References

[1] **Taub, A.**, Similarity Solutions for Perfect Fluids, *Comm. Math. Phys.* **21:** 1-20 (1971).
[2] **Ernst, F.J.**, New Formulation of Stationary Gravitational Fields, *Phys. Rev.* **167:** 1175-1178 (1968).
[3] **Halilsoy, M.**, Similarity Solution for Ernst Equation with Electromagnetic Fields, *Lett. Nuovo Cimento* **37:** 231-235 (1983).
[4] **Lewis, T.**, Some Special Solutions of the Equations of Axially Symmetric Gravitational Fields, *Proc. Roy. Soc. (London)*, A **136:** 176-190 (1932).
[5] **van Stockum, W.J.**, The Gravitational Field of a Distribution of Particles Rotating About an Axis of Symmetry, *Proc. Roy. Soc. (Edinburg)*, **57:** 135-145 (1937).
[6] **Halilsoy, M.**, Reduction of Certain Classes of Partial Differential Equations, *Journal of Engg and App. Sciences*, **3:** 125-135 (1986).
[7] **Eells, J.** and **Sampson, J.H.**, Harmonic Mapping of Riemannian Manifolds, *Amer. J. Math.* **86:** 109-160 (1964).
[8] **Levi-Civita, T.**, Ds^2 Einsteiniani in campi Newtoniani, *Rend. Accad. Lincei* **26:** 308-318 (1917).
[9] **Mazur, P.O.**, Spinning Cosmic Strings and Quantization of Energy, *Phys. Rev. Lett.* **57:** 929-931 (1986).
[10] **Newman, E.** and **Penrose, R.**, An Approach to Gravitational Radiation by a Method of Spin Coefficients, *J. Math. Phys.* **3:** 566-577 (1962).
[11] **Kerr, R.**, Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics, *Phys. Rev. Lett.* **11:** 237-239 (1963).

التحليل التماثـــلي للمجـــال التجاذبي

مصـــطفى خليـــل صـــوى
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تمت مراجعـة مجال فراغ التـماثـل المحوري الثابت في الفضاء - الزمن والتي هي خاصة التماثل . يصمم الحل الذي حصلنا عليه حلول لويس وفان ستوكوم .

وقـد وجـدت الجهود ومركبات الرابوغ الفراغي لتنسر ريمان حيث وصلنا إلى أن حل المجال الدوراني بصيغة غير مقيدة متعارض مع مفهوم التشابه الذاتي .