Journal of Mathematical Analysis and Applications **231,** 68–75 (1999)
Article ID jmaa.1998.6142, available online at http://www.idealibrary.com on IDEAL®

Random Fixed Points of 1-Set Contractive Random Maps in Frechet Spaces

Naseer Shahzad

Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

and

Liaqat Ali Khan*

Department of Mathematics, King Abdul Aziz University, P.O. Box 9028, Jeddah 21413, Saudi Arabia

Submitted by N. H. Bingham

Received November 25, 1997

A random fixed point theorem for more general 1-set-contractive random operators in Fréchet spaces is proved. As applications, some random fixed point theorems for nonexpansive, semicontractive, and LANE random operators are also derived. Thus the results of Itoh [*J. Math. Anal. Appl.* **67** (1979), 261–273], Lin [*Proc. Amer. Math. Soc.* **123** (1995), 1167–1176; *Proc. Amer. Math. Soc.* **103** (1988), 1129–1135], Shahzad [*J. Math. Anal. Appl.* **203** (1996), 712–718], Tan and Yuan [*Stochastic Anal. Appl.* **15** (1997), 103–123], and Xu [*Proc. Amer. Math. Soc.* **110** (1990), 495–500], are extended.

Key Words: random operator; 1-set-contractive map; random fixed point; Fréchet space

1. INTRODUCTION

Probabilistic functional analysis has emerged as one of the important mathematical disciplines in view of its role in dealing with probabilistic models in applied problems. The study of random operators forms a central topic in this discipline. The development of a theory of random operators is required for the study of various classes of random equations.

* On leave from Quaid-i-Azam University, Islamabad, Pakistan.

0022-247X/99 $30.00

The study of random fixed point theorems was initiated by the Prague school of probabilists in the 1950s. The interest in the generalizations of random fixed point theorems from self-maps to nonself-maps has been revived after the papers by Sehgal and Waters [19], Sehgal and Singh [20], Papageorgiou [12, 13], Lin [9, 10], Xu [25], Tan and Yuan [22, 23], Beg and Shahzad [2−4] and Shahzad [21]. In particular, Lin [9], Tan and Yuan [22], and Beg and Shahzad [4] studied random fixed points of 1-set-contractive maps. The class of 1-set-contractive random maps includes condensing, nonexpansive, and other interesting random maps such as locally almost nonexpansive (LANE) maps, semicontractive maps, and the sum of a nonexpansive and a completely continuous random map, etc. Shahzad [21] obtained a random fixed point theorem for more general 1-set-contractive nonself maps in Banach spaces which unifies and extends the work of Itoh [7], Lin [9, 10] and Xu [25]. The aim of this paper is to further extend the main result of Shahzad [21] to Fréchet spaces. We apply our main result to derive some random fixed point theorems for nonexpansive, semicontractive, or LANE random maps.

2. PRELIMINARIES

Let $(\Omega, \mathscr{A})$ be a measurable space with $\mathscr{A}$ a sigma algebra of subsets of Ω. Let S be a nonempty subset of a Fréchet space X. A mapping G: $\Omega \to 2^s \setminus \{\phi\}$ is called measurable if, for each open subset B of S, $G^{-1}(B) \in \mathscr{A}$, where 2^s is the family of all subsets of X and $G^{-1}(B) = \{\omega \in \Omega: G(\omega) \cap B \neq \phi\}$. A mapping ξ: $\Omega \to S$ is said to be a measurable selector of a measurable mapping G: $\Omega \to S^s \setminus \{\phi\}$ if ξ is measurable and, for any $\omega \in \Omega$, $\xi(\omega) \in G(\omega)$. A mapping f: $\Omega \times S \to X$ is called a random operator if, for each fixed $x \in S$, the map $f(\cdot, x)$: $\Omega \to X$ is measurable. A measurable map ξ: $\Omega \to S$ is a random fixed point of a random operator f: $\Omega \times S \to X$ if $\xi(\omega) = f(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

We know that a locally convex topological vector space (always assumed Hausdorff) X is metrizable if and only if X has a countable base of absolutely convex neighbourhoods of zero or, equivalently, X has a countable family of seminorms $\{p_n\}$ that generates the locally convex topology on X. We can always assume that $p_n \le p_{n+1}$, $n \ge 1$. A function d: $X \times X \to \mathbb{R}^+ \cup \{0\}$ given by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{c_n p_n(x-y)}{1 + p_n(x-y)},$$

for $x \in X$, where $c_n > 0$ and $\sum_{n=1}^{\infty} c_n < \infty$, defines a metric on X.

Let B be a nonempty bounded subset of a Fréchet space X. Let α be the Kuratowski measure of noncompactness, that is, $[\alpha(B)](p_n) = \inf\{c > 0$: B can be covered by a finite number of sets whose diameters with respect to $p_n \leq c\}$ (cf. Sadovskii [18]). Let S be a nonempty subset of X. A mapping f: $S \to X$ is called condensing if f is continuous and, for any nonempty bounded subset B of S with $[\alpha(B)](p_n) > 0$, $[\alpha(f(B))](p_n) < [\alpha(B)](p_n)$ for all $n \geq 1$. If for each nonempty bounded subset B of S, $[\alpha(f(B))](p_n) \leq [\alpha(B)](p_n)$ for all $n \geq 1$, then a continuous map f: $S \to X$ is called a 1-set-contractive map. A map f: $S \to X$ is said to be demiclosed at $y \in X$ if, for any sequence $\{x_j\}$ in S, the condition $x_j \to x$ in S weakly and $f(x_j) \to y$ strongly imply $f(x) = y$. A random operator f: $\Omega \times S \to X$ is continuous (1-set-contractive, condensing, etc.) if the map $f(\omega, \cdot)$: $S \to X$ is so for each $\omega \in \Omega$. We denote by I the identity mapping of X. A random operator f: $\Omega \times S \to X$ is said to be weakly inward if, for any $\omega \in \Omega$, $f(\omega, x) \in c\ell(I_s(x))$ for all $x \in S$, where $I_s(x) = \{z \in X$: $z = x + a(y - x)$ for some $y \in S$ and $a \geq 0\}$. When S has a nonempty interior, a random operator f: $\Omega \times S \to X$ is said to satisfy the Leray–Schauder condition if there is a point z in $\mathrm{int}(S)$, the interior of S, (depending on ω) such that

$$f(\omega, y) - x \neq m(y - x), \tag{1}$$

for all $y \in \partial S$, the boundary of S, and $m > 1$. Throughout this paper, we shall assume that the interior of S is nonempty whenever f satisfies the Leray–Schauder condition on S.

Let S be a nonempty bounded subset of a normed space X. A mapping f: $S \to X$ is called nonexpansive if $\|f(x) - f(y)\| \leq \|x - y\|$ for each $x, y \in S$. A continuous mapping f: $S \to X$ is called LANE (locally almost nonexpansive) if given $x \in S$ and $\varepsilon > 0$, there exists a weak neighbourhood N_x of x in S (depending also on ε) such that $u, v \in N_x$, $\|f(u) - f(v)\| \leq \|u - v\| + \varepsilon$. A mapping f: $S \to X$ is completely continuous if, for any $\{x_j\}$ in S such that $x_j \to x_0$ weakly in S, $f(x_j) \to f(x_0)$ strongly in X as $j \to \infty$. A continuous mapping f: $S \to X$ is called semicontractive if there exists a map V: $S \times S \to X$ such that $f(x) = V(x, x)$ for x in S, and for each fixed x in S, $V(\cdot, x)$ is nonexpansive from S to X and $V(x, \cdot)$ is completely continuous from S to X uniformly for x in S, (that is, if v_j converges weakly to x in S and $\{u_j\}$ is a sequence in S, then $V(u_j, v_j) - V(u_j, x) \to 0$ strongly in S). A random operator f: $\Omega \times S \to X$ is said to be nonexpansive (LANE, completely continuous, semicontractive, etc.) if, for each $\omega \in \Omega$, $f(\omega, \cdot)$ is nonexpansive (LANE, completely continuous, semicontractive, etc.).

3. MAIN RESULTS

In what follows we need the following random fixed point theorem, which is a single-valued version of Beg and Shahzad [1, Theorem 3.1].

THEOREM 3.1. *Let S be a nonempty separable closed subset of a complete metric space X and let D be a countable dense subset of S. If a continuous random operator $f: \Omega \times S \to X$ satisfies the conditions,*

(a) $G(\omega) = \{x \in S: x = f(\omega, x)\}$ *is nonempty for each* $\omega \in \Omega$; *and*

(b) *each sequence* $\{x_j\} \subseteq D$ *with* $d(x_j, f(\omega, x_j)) \to 0$ *has a convergent subsequence,*

then f has a random fixed point.

THEOREM 3.2. *Let S be a nonempty weakly compact convex subset of a separable Fréchet space X and let $f: \Omega \times S \to X$ be a condensing random operator that is either* (i) *weakly inward or* (ii) *satisfies the Leray–Schauder condition. If, for any $\omega \in \Omega$, $f(\omega, S)$ is bounded, then f has a random fixed point.*

Proof. Define a mapping $G: \Omega \to 2^S$ by

$$G(\omega) = \{x \in S: x = f(\omega, x)\},$$

then, by Polewczak [15, Theorem II.9] and Reich [16, Theorem 3.5], in both cases, $G(\omega)$ is nonempty for each $\omega \in \Omega$. Now, let $d(x_j, f(\omega, x_j)) \to 0$ for any $\omega \in \Omega$, $\{x_j\} \subset D$, where D is a countable dense subset of S. Let $B = \{x_j: j \geq 1\}$ and $\gamma_n = [\alpha(f(\omega, B))](p_n)$ for $n \geq 1$. Then, for any $\varepsilon > 0$, there exists a finite number of sets $B_1, B_2, \dots, B_k$ of X each of diameter less than or equal to $(\gamma_n + \varepsilon/2)$ such that $f(\omega, B) \subseteq \bigcup_{i=1}^{k} B_i$. Let, for each i, A_i be a $\varepsilon/2$ neighbourhood of B_i and choose an integer N such that $d(x_j, f(\omega, x_j)) < \varepsilon/2$ for all $j \geq N$. Then $\{x_j: j \geq N\} \subseteq \bigcup_{i=1}^{k} A_i$ and the diameter of each A_i is less than or equal to $\gamma_n + \varepsilon$. Because ε is arbitrary, it follows that

$$\begin{aligned}[\alpha(B)](p_n) &= \left[\alpha\{x_j: j \geq N\}\right](p_n) \leq \gamma_n \\ &= \left[\alpha(f(\omega, B))\right](p_n),\end{aligned}$$

for each $n \geq 1$. Thus $[\alpha(B)](p_n) = 0$ for each $n \geq 1$. Because S, as a weakly compact set, is a complete subset of X, the set $B = \{x_j: j \geq 1\}$ is precompact. Thus conditions (a) and (b) of Theorem 3.1 are satisfied. Hence f has a random fixed point.

THEOREM 3.3. *Let S be a nonempty weakly compact convex subset of a separable Fréchet space and let $f: \Omega \times S \to X$ be a* 1*-set-contractive random* *operator that is either* (i) *weakly inward or* (*ii*) *satisfies the Leray–Schauder condition. If, for any* $\omega \in \Omega$, $f(\omega, S)$ *is bounded and* $I - f(\omega, \cdot)$ *is demiclosed at zero, then f has a random fixed point.*

Proof. Let f satisfy (i). Then take an element v in S and a sequence $\{k_n\}$ of real numbers with $0 < k_n < 1$ and $k_n \to 0$ as $n \to \infty$. For each n, consider the mapping f_n: $\Omega \times S \to X$ defined by $f_n(\omega, x) = k_n v + (1 - k_n) f(\omega, x)$. The properties of the measure of noncompactness α and f imply that, for each n, f_n is a weakly inward $(1 - k_n)$-set-contractive random operator. Because, for each $\omega \in \Omega$, $f(\omega, S)$ is bounded, $f_n(\omega, S)$ is bounded too. Thus, for each n, f_n has a random fixed point ξ_n: $\Omega \to S$ by Theorem 3.2(i). For each n, define G_n: $\Omega \to WK(S)$ by

$$G_n(\omega) = \mathscr{w}\text{-}c\ell\{\xi_i(\omega);\, i \geq n\},$$

where $WK(S)$ is the family of all nonempty weakly compact subsets of S and $\mathscr{w}\text{-}c\ell$ denotes the weak closure. Define G: $\Omega \to WK(S)$ by $G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega)$. Because S is weakly compact and X is separable, the weak topology on S is a metric topology (cf. Rudin [17, p. 86]). By Himmelberg [6, Theorem 4.1], G is $\mathscr{w}$-measurable (that is, G is measurable with respect to the weak topology on S). Thus, by the Kuratowski and Ryll-Nardzewski selection theorem [8], there exists a $\mathscr{w}$-measurable selector ξ of G. For each x^* in X^*, the dual space of X, the numerically valued function $x^*(\xi(\cdot))$ is measurable. Because X is separable, ξ is measurable (cf. Thomas [24, Theorem 1]). This ξ is the desired random fixed point of f. Indeed, for any fixed $\omega \in \Omega$, some subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ converges weakly to $\xi(\omega)$. Because S and $f(\omega, S)$ are bounded, $k_m \to 0$ as $m \to \infty$, and

$$\xi_m(\omega) - f(\omega, \xi_m(\omega)) = k_m\{v - f(\omega, \xi_m(\omega))\},$$

$\xi_m(\omega) - f(\omega, \xi_m(\omega))$ converges strongly to 0 (cf. Rudin [17, p. 22]). Because $I - f(\omega, \cdot)$ is demiclosed at zero, we have $\xi(\omega) = f(\omega, \xi(\omega))$.

Suppose now condition (ii) is satisfied. Let $z \in \operatorname{int}(S)$ satisfy (1). Take a sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $k_n \to 0$ as $n \to \infty$. Then one easily sees that the mapping f_n: $\Omega \times S \to X$ defined by $f_n(\omega, x) = k_n z + (1 - k_n) f(\omega, x)$ is a $(1 - k_n)$-set-contractive random operator that satisfies the Leray–Schauder condition. Then, by Theorem 3.2(ii), f_n has a random fixed point ξ_n. Let $G_n(\omega) = \mathscr{w}\text{-}c\ell\{\xi_i(\omega) \colon i \geq n\}$ and $G(\omega) = \bigcap_{n=1}^{\infty} G_n(\omega)$. Then, as in the first part of the proof, G is weakly measurable and has a measurable selector ξ. This ξ is clearly a random fixed point of f.

The following is an easy consequence of Theorem 3.3.

COROLLARY 3.4. *Let S be a nonempty closed bounded convex subset of a separable reflexive Banach space X and let $f\colon \Omega \times S \to X$ be a 1-set-contractive random operator that is either* (i) *weakly inward or* (ii) *satisfies the Leray–Schauder condition. If, for each $\omega \in \Omega$, $f(\omega, S)$ is bounded and $I - f(\omega, \cdot)$ is demiclosed at zero, then f has a random fixed point.*

COROLLARY 3.5. *Let S be a nonempty closed bounded convex subset of a separable uniformly convex Banach space X, let $g\colon \Omega \times S \to X$ be a nonexpansive random operator, and let $h\colon \Omega \times S \to X$ be a completely continuous random operator. If the random operator $f = g + h\colon \Omega \times S \to X$ is either* (i) *weakly inward or* (ii) *satisfies the Leray–Schauder condition, then f has a random fixed point.*

Proof. Because X is uniformly convex (and thus reflexive) and h is completely continuous, $\alpha(h(\omega, A)) = 0$ for each subset A of S and for each $\omega \in \Omega$. Hence $f = g + h\colon \Omega \times S \to X$ is a 1-set-contractive random operator. Further, as in Shahzad [21, proof of Theorem 3.6], $I - f(\omega, \cdot)$ is demiclosed at zero for each $\omega \in \Omega$. Because S is weakly compact, f has a random fixed point from Theorem 3.3.

COROLLARY 3.6. *Let S be a nonempty closed bounded convex subset of a separable uniformly convex Banach space X, let $g\colon \Omega \times S \to X$ be a LANE random operator, and let $h\colon \Omega \times S \to X$ be a completely continuous random operator. If the random operator $f = g + h\colon \Omega \times S \to X$ is either* (i) *weakly inward or* (ii) *satisfies the Leray–Schauder condition, then f has a random fixed point.*

Proof. Because X is reflexive, $h\colon \Omega \times S \to X$ is completely continuous, and $g\colon \Omega \times S \to X$ is a LANE random operator, $f = g + h\colon \Omega \times S \to X$ is 1-set-contractive (cf. Nussbaum [11]). Also, for each $\omega \in \Omega$, $I - f(\omega, \cdot)$ is demiclosed at zero as in Shahzad [21, proof of Theorem 3.10]. Hence the corollary follows from Theorem 3.3.

COROLLARY 3.7. *Let S be a nonempty closed bounded convex subset of a separable uniformly convex Banach space X and let $f\colon \Omega \times S \to X$ be a continuous semicontractive random operator that is either* (i) *weakly inward or* (ii) *satisfies the Leray–Schauder condition. Then f has a random fixed point.*

Proof. By Petryshyn [14, Lemma 3.2 and p. 338], f is 1-set-contractive and, by Browder [5, Theorem 3] $I - f(\omega, \cdot)$ is demiclosed at zero for each $\omega \in \Omega$. Theorem 3.3 further implies that f has a random fixed point.

Remark 3.8. 1. Theorem 2.1 of Lin [9] and Theorems 3.3 and 3.4 of Tan and Yuan [22] can be viewed as special cases of Theorem 3.3.

2. The theorems and corollaries in this paper extend Section 2 of Lin [9] (that is, Theorem 2.1–Corollary 2.2) to weakly inward maps.

3. Corollaries 3.4–3.7 were originally proved by Shahzad [21] with the separability condition on S.

4. Corollary 3.5 generalizes Theorems 2.6 and 3.6 of Itoh [7], Theorem 6(ii) of Lin [10], and Theorem 4 of Xu [25].

REFERENCES

1. I. Beg and N. Shahzad, A general fixed point theorem for a class of continuous random operator, *New Zealand J. Math.* **26** (1997), 21–24.
2. I. Beg and N. Shahzad, Random fixed points of weakly inward operators in conical shells, *J. Appl. Math. Stoch. Anal.* **8** (1995), 261–264.
3. I. Beg and N. Shahzad, Random fixed point theorems for multivalued inward random operators on Banach spaces, *Adv. Math. Sci. Appl.* **5**(1) (1995), 31–37.
4. I. Beg and N. Shahzad, Applications of the proximity map to random fixed point theorems in Hilbert spaces, *J. Math. Anal. Appl.* **196** (1995), 606–613.
5. F. E. Browder, Semicontractive and semiaccretive nonlinear mappings in Banach spaces, *Bull. Amer. Math. Soc.* **74** (1968), 660–665.
6. C. J. Himmelberg, Measurable relations, *Fund. Math.* **87** (1975), 53–72.
7. S. Itoh, Random fixed point theorems with an application to random differential equations in Banach spaces, *J. Math. Anal. Appl.* **67** (1979), 261–273.
8. K. Kuratowski and C. Ryll-Nardzewski, A general theorem on selectors, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **13** (1965), 397–403.
9. T. C. Lin, Random approximations and random fixed point theorems for continuous 1-set-contractive random maps, *Proc. Amer. Math. Soc.* **123** (1995), 1167–1176.
10. T. C. Lin, Random approximations and random fixed point theorems for nonself maps, *Proc. Amer. Math. Soc.* **103** (1988), 1129–1135.
11. R. D. Nussbaum, "The Fixed Point Index and Fixed Point Theorems for k-Set-Contractions," Ph.D. thesis, University of Chicago, 1969.
12. N. S. Papageorgiou, Random fixed point theorems for measurable multifunctions in Banach spaces, *Proc. Amer. Math. Soc.* **97** (1986), 507–514.
13. N. S. Papageorgiou, Random fixed points and random differential inclusions, *Internat. J. Math. Math. Sci.* **11** (1988), 551–560.
14. W. V. Petryshyn, Fixed point theorems for various classes of 1-set-contractive and 1-ball-contractive mappings in Banach spaces, *Trans. Amer. Math. Soc.* **182** (1973), 323–352.
15. J. Polewczak, Ordinary differential equations on closed subsets of Fréchet spaces with applications to fixed point theorems, *Proc. Amer. Math. Soc.* **107** (1989), 1005–1012.
16. S. Reich, Fixed points in locally convex spaces, *Math. Z.* **125** (1972), 17–31.
17. W. Rudin, "Functional Analysis," McGraw-Hill, New York, 1973.
18. B. N. Sadovskii, Limit-compact and condensing operators, *Russian Math. Survey* **27** (1972), 85–155.
19. V. M. Sehgal and C. Waters, Some random fixed point theorems for condensing operators, *Proc. Amer. Math. Soc.* **90** (1984), 425–429.
20. V. M. Sehgal and S. P. Singh, On random approximations and a random fixed point theorem for set-valued mappings, *Proc. Amer. Math. Soc.* **95** (1985), 91–94.

21. N. Shahzad, Random fixed point theorems for various classes of 1-set-contractive maps in Banach spaces, *J. Math. Anal. Appl.* **203** (1996), 712–718.
22. K. K. Tan and X. Z. Yuan, Random fixed point theorems and approximation, *Stochastic Anal. Appl.* **15** (1997), 103–123.
23. K. K. Tan and X. Z.Yuan, Random fixed point theorems and approximation in cones, *J. Math. Anal. Appl.* **185** (1994), 378–390.
24. G. E. F. Thomas, Integration on functions with values in locally convex Suslin spaces, *Trans. Amer. Math. Soc.* **212** (1975), 61–81.
25. H. K. Xu, Some random fixed point theorems for condensing and nonexpansive operators, *Proc. Amer. Math. Soc.* **110** (1990), 495–500.